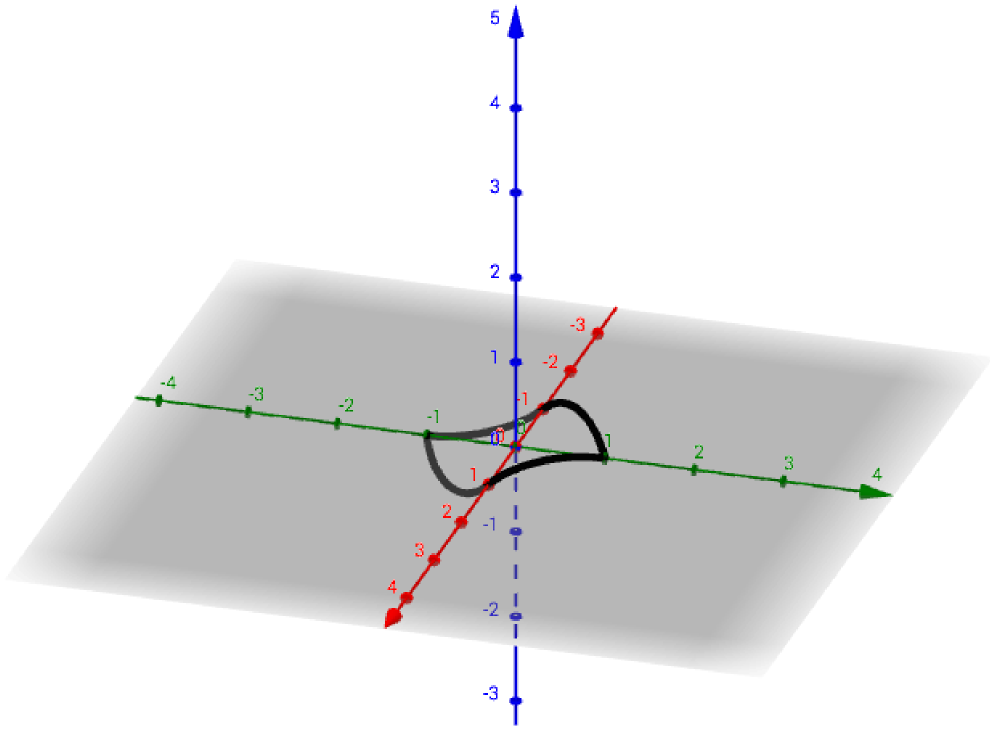
Posted by: **Dave Richeson** | July 5, 2016

Sugihara’s Circle/Square Optical Illusion

Kokichi Sugihara created a video called Ambiguous Optical Illusion: Rectangles and Circles. (https://www.youtube.com/watch?v=oWfFco7K9v8) In it he shows a variety of 3-dimensional objects that look like one shape when viewed from the front but look like a different shape in the mirror behind it.

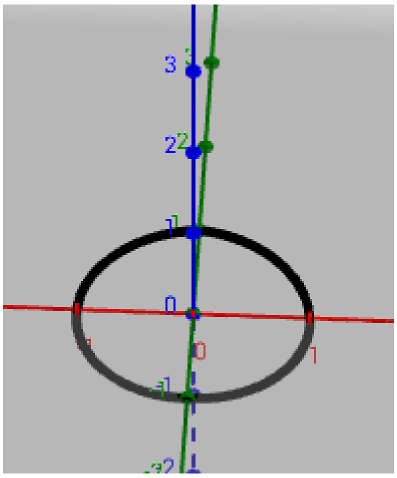
In this blog post we show how he achieved the effect. For simplicity, we will show how he made a shape that looks like a circular cylinder from the front and a square cylinder in the mirror.

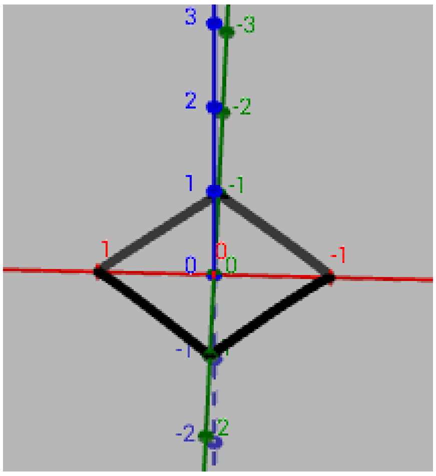
The following applet shows our final product (clicking the image links to the GeoGebra applet (http://ggbm.at/EdSp6X76)). It is a closed curve that represents the top rim of Sugihara’s shape. You can rotate the axes with your mouse. If you view the coordinate system with the positive green and blue axes lined up (1 with 1, 2 with 2, and so on), the curve will look like the unit circle in the green-red plane. If you drag the image so that the positive blue axis lines up with the negative green axis (1 with -1, 2 with -2, and so on), it will look like you are viewing a square (oriented as a diamond) in the green-red plane.



(http://ggbm.at/EdSp6X76)

Here are screenshots showing the two views.

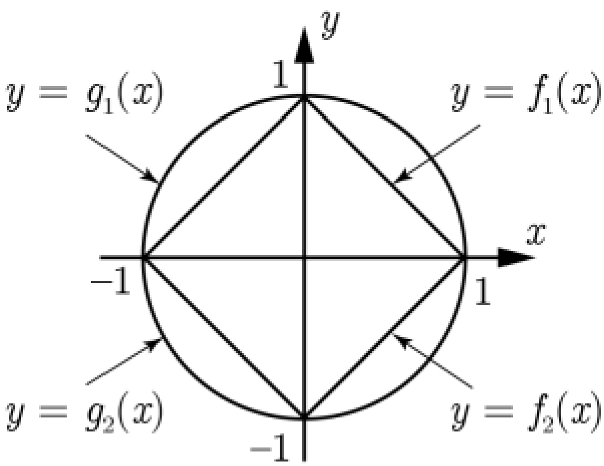




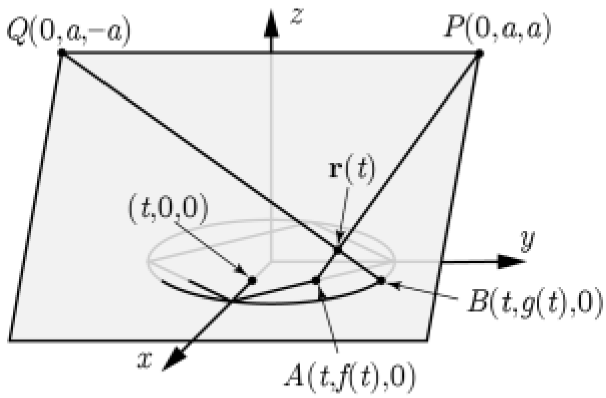
How does it work? It is all about perspective.

To set this up mathematically, we imagine two viewers in 3-dimensional space. One viewer is at $P(0, a, a)$ and the other is at $Q(0, -a, a)$ (in the video this second viewer is you, in the mirror). They are looking down on a curve $\mathbf{r}(t)$. However, from their vantage points it looks like they are seeing two different curves in the xy-plane: $\langle u, f(u), 0\rangle$ and $\langle u, g(u), 0\rangle$, respectively.

In our example the two observed curves are the unit circle and the square passing through the points $(\pm 1, \pm 1, 0)$, as shown in the xy-plane below. We will have to break each of these shapes into two different curves, so we'll have $f_1(u) = 1 - |u|$, $f_2(u) = |u| - 1$, $g_1(u) = \sqrt{1 - u^2}$, and $g_2(u) = -\sqrt{1 - u^2}$. Also, we could choose a to be some suitably large number greater than 1, but in fact, as we will see, taking the limit as a tends to infinity produces a lovely final expression. For now we will continue to work in generalities and will wait to insert these specifics later.



Our aim is now to define $\mathbf{r}(t)$. Let's fix t, and let $A(t, f(t), 0)$ and $B(t, g(t), 0)$ be two points on the curves in the xy-plane. In order for the person at P to view her shape, $\mathbf{r}(t)$ must lie on the line AP (see figure below). Likewise, for the person at Q to see his shape, $\mathbf{r}(t)$ must lie on the line BQ. Thus, $\mathbf{r}(t)$ must be the point of intersection of lines AP and BQ. (We know that the lines are not skew because they lie in the plane containing the points P, Q, and $(t, 0, 0)$, and for appropriate choices of f, g, and a the lines intersect and the point of intersection is below $z = a$.)



It is straightforward to show that $\mathbf{r}_{AP}(s)=\langle ts,(f(t)-a)s+a,-as+a\rangle$ is a parametrization of the line AP and $\mathbf{r}_{BQ}(s)=\langle ts,(g(t)+a)s-a,-as+a\rangle$ is a parametrization of BQ. A little algebra shows shows that their point of intersection is

$$\left(\frac{2at}{g(t)-f(t)+2a},\frac{af(t)+ag(t)}{g(t)-f(t)+2a},\frac{ag(t)-af(t)}{g(t)-f(t)+2a}\right),$$

and thus our desired curve is

$$\mathbf{r}(t)=\tfrac{a}{g(t)-f(t)+2a}\cdot\langle 2t,f(t)+g(t),g(t)-f(t)\rangle.$$

Because a is a large value, we can take the limit as a goes to infinity. This yields the elegant expression

$$\mathbf{r}(t)=\langle t,\tfrac{1}{2}(f(t)+g(t)),\tfrac{1}{2}(g(t)-f(t))\rangle.$$

We may now plug in our functions. The portion of our curve with nonnegative y-coordinates is given by

$$\mathbf{r}_1(t)=\langle t,\tfrac{1}{2}(1-|t|+\sqrt{1-t^2}),\tfrac{1}{2}(\sqrt{1-t^2}+|t|-1)\rangle$$

for $-1\le t\le 1$, and the other half by

$$\mathbf{r}_2(t)=\langle t,\tfrac{1}{2}(|t|-1-\sqrt{1-t^2}),(1-\sqrt{1-t^2}-|t|)\rangle$$

for $-1\le t\le 1$. This is the curve shown in the applet.

[Note (7/6/16): The original version of this blog post didn't have the limit as a tends to infinity. I noticed that trick after publishing the post.]