Optimal performance for Tesla transformers

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The previous work related to finding improved performance for Tesla transformers is shortly reviewed. The possibilities to reach the optimal working point by modifying the main components are discussed from a practical standpoint. A methodology for maximizing the secondary voltage by regulating the tuning ratio T and the coupling coefficient is examined in particular. It is shown that its results are valid only if primary and secondary inductor values remain unchanged, and the secondary capacitor value is decreased. All in all, the best improvement from the typical condition of $T=1$ increases the secondary voltage of only 18% and requires tide coupling. This, in turn, imposes severe engineering problems to avoid dielectric breakdown between the primary and secondary coils, and makes the practical utility of this result someway questionable. In a real Tesla transformer, the most practical mean to perform tuning is to move the tap feeding the primary coil, rather than rewinding the secondary coil or redesigning the secondary top terminal. The resonant circuits are not undamped and it is crucial to reach the maximum voltage at the secondary in the shortest time, to minimize losses. It is shown that, in order to achieve optimal performance, a better strategy is to tune the primary coil to achieve $T=1$ and then to increase the coupling coefficient as much as possible, aiming at one of the values selected from a given table. [DOI: 10.1063/1.1498905]

I. INTRODUCTION

The typical Tesla transformer is composed of two circuits. The primary circuit consists of a high-voltage capacitor that is discharged through a switching device (e.g., a spark gap) into a low-inductance coil. The secondary circuit simply features an air-wound coil with one side grounded and the other side connected to a terminal (usually a sphere or a toroid). If the two coils are magnetically coupled, every discharge of the primary capacitor generates a magnified voltage on the secondary coil.

The working point of the Tesla transformer is influenced by the values of capacitance and inductance of primary and secondary circuits, together with the amount of coupling between them. Attempting to maximize its efficiency is not a trivial task, as these parameters all have a nonlinear effect on the transformer tuning and its magnification.

The purpose of this article is to provide a short review of the previous work related to the optimization of Tesla transformers, showing that different threads converge to consistent results even if there is not a consensus on the definition of the targeted optimum. More, the possibilities to reach the desired working point are discussed from a practical point of view, together with the magnitude of the obtainable improvement.

II. PREVIOUS WORK

Conditions required to achieve the maximum voltage at the secondary circuit of a Tesla transformer were first pointed out by Drude[1] and consisted of a unitary tuning ratio and a coupling coefficient of 0.6. From that point, the search for an optimal working point has evolved along two axes.

Targeting the maximum output voltage, Reed[2] has observed that an 18% increase can be obtained by using a tuning ratio less than unity and a suitable amount of coupling. His work has been generalized by Phung *et al.*,[3] providing a set of equations in order to calculate all tuning ratio and coupling coefficient pairs that achieve a (local) maximum output.

Following an alternative track, Finkelstein[4] has identified the general conditions required for a complete energy transfer from the primary to the secondary circuits: in all cases, a unitary tuning ratio is required. The Drude's conditions achieve complete energy transfer in the least time, but other values of coupling coefficient can be used as well, while the transfer completion is simply moved to a later time instant. Finkelstein's work was continued and extended to three coupled resonance circuits by Bieniosek[5] and eventually generalized to any number of circuits by de Queiroz.[6,7]

III. AIR-COUPLED RESONANT CIRCUITS

Operation of the Tesla transformer can be regarded as that of two inductively air-coupled, damped resonant circuits (Fig. 1). The primary circuit is formed when the spark gap conducts and connects in series the primary capacitor C_1, the primary coil L_1, and its equivalent resistance R_1. The secondary circuit is formed by the series of the secondary coil L_2 with its equivalent resistance R_2 and with the top toroid C_2. The loop is closed through ground, as the secondary coil base is grounded and the top toroid exhibits a lumped capacity with respect to ground also. The primary and secondary coils are inductively coupled with each other with mutual inductance M.

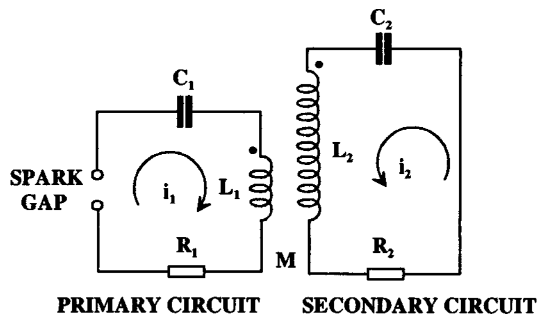

FIG. 1. Inductively coupled primary and secondary circuits in a Tesla transformer.

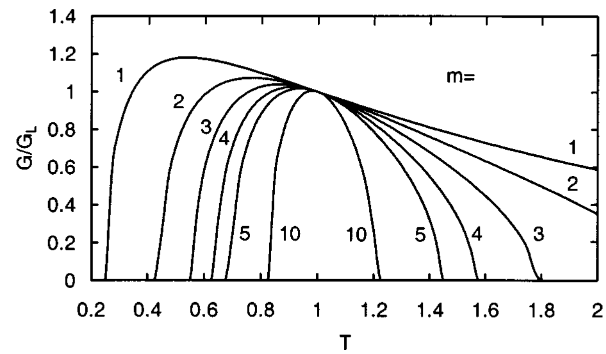

FIG. 2. G/G_L gain vs tuning ratio T for different m values.

According to the first Kirchoff law, the sum of the voltages around a closed circuit is zero, therefore,[8,9]

$$R_1 i_1 + \frac{1}{C_1}\int i_1\,dt + L_1\frac{di_1}{dt} + M\frac{di_2}{dt} = 0, \tag{1}$$

$$R_2 i_2 + \frac{1}{C_2}\int i_2\,dt + L_2\frac{di_2}{dt} + M\frac{di_1}{dt} = 0. \tag{2}$$

Solutions in a closed form for voltage v_2 developed on capacitor C_2 can be found only for the ideal case of no damping ($R_1 = R_2 = 0$) as[3]

$$v_2(t) = \frac{2kV_1}{\sqrt{(1-T)^2+4k^2T}}\sqrt{\frac{L_2}{L_1}}\sin\left(\frac{w_2+w_1}{2}t\right) \times \sin\left(\frac{w_2-w_1}{2}t\right), \tag{3}$$

where

$$k = \frac{M}{\sqrt{L_1L_2}}, \tag{4}$$

$$\omega_i = \frac{1}{\sqrt{L_iC_i}}, \quad i = 1,2, \tag{5}$$

$$T = \frac{\omega_1^2}{\omega_2^2} = \frac{L_2C_2}{L_1C_1}, \tag{6}$$

$$w_1 = \omega_2\sqrt{\frac{(1+T)-\sqrt{(1-T)^2+4k^2T}}{2(1-k^2)}},$$
$$w_2 = \omega_2\sqrt{\frac{(1+T)+\sqrt{(1-T)^2+4k^2T}}{2(1-k^2)}}. \tag{7}$$

Here k is the coupling coefficient ($0<k<1$), while ω_1 and ω_2 are, respectively, the angular resonance frequencies of the uncoupled primary and secondary circuits (also called open-circuit resonances).

The tuning ratio, indicated by T, is defined as the square of the ratio of the uncoupled resonance frequencies, while V_1 is the initial voltage across C_1. More, w_1 and w_2 are the angular resonance frequencies of the primary and secondary circuits when coupled. The physical constraints on the values of k and T ensure that w_1 and w_2 are always real. Note that $w_2 > w_1$ is also assumed.

Equation (3) shows that the secondary voltage is a high frequency oscillation $(w_2+w_1)/2$ which is amplitude modulated by another low frequency oscillation $(w_2-w_1)/2$.

IV. CONDITIONS FOR MAXIMUM VOLTAGE GAIN

An obvious way to optimize a Tesla transformer design is to aim for developing the maximum achievable secondary voltage. From Eq. (3), the maximum voltage gain is

$$G = \left.\frac{v_2(t)}{V_1}\right|_{\max} = \frac{2k}{\sqrt{(1-T)^2+4k^2T}}G_L, \tag{8}$$

where

$$G_L = \sqrt{\frac{L_2}{L_1}}. \tag{9}$$

The gain G from Eq. (8) can be achieved only if both the sine terms in Eq. (3) are equal to ± 1 simultaneously, that is, only if

$$\frac{w_2+w_1}{2}t = \frac{\pi}{2} + m\pi \quad \text{and} \quad \frac{w_2-w_1}{2}t = \frac{\pi}{2} + n\pi, \tag{10}$$

where n and m are positive or negative integers. Without losing generality, n can be set to zero, therefore, changing the requirement to

$$\frac{w_2}{w_1} = \frac{1+m}{m}. \tag{11}$$

Substituting Eq. (7) into Eq. (11) gives

$$k = \sqrt{\frac{\alpha^2(1+T)^2-(1-T)^2}{4T}}, \tag{12}$$

where

$$\alpha = \frac{1+2m}{1+2m+2m^2}. \tag{13}$$

k can now be eliminated from Eq. (8) giving[3]

$$G = \left.\frac{v_2(t)}{V_1}\right|_{\max} = \sqrt{\frac{\alpha^2(1+T)^2-(1-T)^2}{\alpha^2T(1+T)^2}}G_L. \tag{14}$$

The above result has been previously investigated[2,3] striving for maximizing the G/G_L ratio and pointing out that a value higher than 1 can be obtained for tuning ratios different than unity. As Fig. 2 shows, this ratio has its maximum

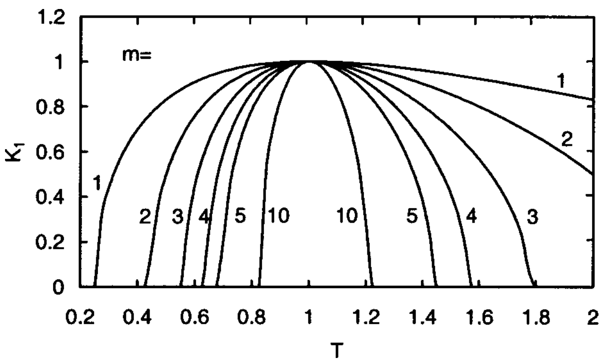

FIG. 3. K_1 gain vs tuning ratio T for different m values.

for $T<1$ depending on the value chosen for m. For instance,[3] if $m=1$ then the maximum value achievable for G/G_L is 1.18, when $T=0.541$ and $k=0.546$.

This analysis results in a maximized $v_2(t)$ peak value supposing that the values of G_L (i.e., L_1 and L_2) and of V_1 remain constant. As the tuning ratio T [Eq. (6)] can be varied by operating on C_1, C_2, L_1, or L_2, it is of practical interest to examine each of these cases separately in terms of the overall voltage gain achieved at the secondary. From Eqs. (3) and (14), the maximum secondary voltage is

$$V_2=v_2(t)|_{\max}=V_1G=V_1\sqrt{\frac{\alpha^2(1+T)^2-(1-T)^2}{\alpha^2T(1+T)^2}}G_L$$
$$=V_1G_T\sqrt{\frac{L_2}{L_1}}, \tag{15}$$

where

$$G_T=\frac{G}{G_L}=\sqrt{\frac{\alpha^2(1+T)^2-(1-T)^2}{\alpha^2T(1+T)^2}}. \tag{16}$$

It has to be noted that choosing a value for m, calculating α from Eq. (13), picking a value for T, and calculating k using Eq. (12), Eq. (11) is satisfied and the sine terms product is maximum (±1).

A. Optimizing L_1 or L_2

From Eqs. (9) and (6), the tuning ratio is

$$T=\frac{L_2C_2}{L_1C_1}=G_L^2\frac{C_2}{C_1}. \tag{17}$$

Optimizing L_1 or L_2, C_1 and C_2 are constant, therefore,

$$G_L=\sqrt{\frac{C_1}{C_2}T}\propto\sqrt{T}. \tag{18}$$

Also as V_1 is constant, from Eqs. (15) and (18) we obtain

$$V_2\propto G_T\sqrt{T}. \tag{19}$$

This means that when varying T by changing L_1 or L_2, the original graph family from Fig. 2 has to be corrected by a factor of $\sqrt{T}$. Therefore, the variable K_1 can be defined as

$$V_2\propto K_1=G_T\sqrt{T}. \tag{20}$$

From Fig. 3 it can be easily seen that the maximum V_2 value is now reached with $T=1$, regardless of the value of m (anyway, m has to be an integer as seen before). In practice, on a real Tesla transformer, the secondary coil cannot be easily modified but the primary coil tap can be moved with no problem. In light of the above considerations, when tuning by moving the primary coil tap, the best performance can be achieved with a tuning ratio $T=1$.

B. Optimizing C_1

When evaluating the benefits of optimizations involving the change of C_1, it is fair to use the same amount of energy E_0 to accumulate the initial charge q_0 on it. Recalling that V_1 is the initial voltage across C_1,

$$E_0=\tfrac{1}{2}C_1V_1^2. \tag{21}$$

From Eqs. (15) and (21), eliminating V_1 we obtain

$$V_2=\sqrt{\frac{2E_0}{C_1}}G_T\sqrt{\frac{L_2}{L_1}}. \tag{22}$$

Substituting Eq. (6) and eliminating C_1 the maximum secondary voltage is

$$V_2=\sqrt{2E_0}\sqrt{\frac{L_1T}{L_2C_2}}G_T\sqrt{\frac{L_2}{L_1}}. \tag{23}$$

As L_1, L_2, C_2, and E_0 are constant, V_2 is now proportional to K_2 defined as

$$V_2\propto K_2=\sqrt{T}G_T. \tag{24}$$

This means that the same results obtained in the previous chapter for K_1 (see Fig. 3) can also be applied in this case. Therefore, when tuning by changing the primary capacitor value, the best performance can be achieved with a tuning ratio $T=1$.

C. Optimizing C_2

Equation (15) is influenced by the value of C_2 only in its G_T term; therefore, the results presented in Fig. 2 apply without any correction when the Tesla transformer optimization is performed by varying the secondary (i.e., top terminal) capacity.

V. CONDITIONS FOR COMPLETE ENERGY TRANSFER

A slightly different approach is to define the Tesla transformer optimal functional mode as one where all of the energy initially present on C_1 gets eventually transferred to C_2, possibly in the shortest amount of time.

Intuitively,[4] a complete transfer of the energy present on C_1 to C_2 implies the developed voltage V_2 to be maximum. As seen previously, this requires Eq. (10) to be satisfied. That can be rearranged as

$$\frac{w_2}{w_1}=\frac{a+2b-1}{a},\quad a,b=1,2,3,\ldots. \tag{25}$$

It can be shown[4] that a further condition required for the whole energy present on C_1 to move to C_2 is

TABLE I. Some of the values of k that ensure complete energy transfer if $T=1$.

a	c	k	Beat	Cycles
1	2	0.600	1	1.0
2	3	0.385	1	1.5
3	4	0.280	1	2.0
4	5	0.220	1	2.5
5	6	0.180	1	3.0
6	7	0.153	1	3.5
7	8	0.133	1	4.0
8	9	0.117	1	4.5
9	10	0.105	1	5.0
10	11	0.095	1	5.5
11	12	0.087	1	6.0
12	13	0.080	1	6.5
1	4	0.882	2	2.0
2	5	0.724	2	2.5
4	7	0.508	2	3.5
5	8	0.438	2	4.0
7	10	0.342	2	5.0
8	11	0.308	2	5.5
10	13	0.257	2	6.5
11	14	0.237	2	7.0
13	16	0.205	2	8.0
14	17	0.192	2	8.5
16	19	0.170	2	9.5
17	20	0.161	2	10.0

$$\omega_1=\omega_2. \tag{26}$$

That is, the tuning ratio value must be $T=1$. Substituting in Eq. (7) we obtain

$$w_1=\frac{\omega_1}{\sqrt{1+k}}, \qquad w_2=\frac{\omega_1}{\sqrt{1-k}}. \tag{27}$$

From Eqs. (25) and (27), the value(s) of k needed to ensure complete energy transfer can be found as

$$\frac{w_2}{w_1}=\frac{a+2b-1}{a}=\frac{\sqrt{1+k}}{\sqrt{1-k}}, \tag{28}$$

that is,

$$k=\frac{c^2-a^2}{c^2+a^2}, \tag{29}$$

where

$$c=a+2b-1. \tag{30}$$

Summarizing, a tuning ratio $T=1$ and a value of k as given by Eq. (29) are sufficient to ensure both a maximum voltage at the secondary and a complete energy transfer from the primary. The choice of a and c (and therefore of k) affects only the position of the time instant when the transfer will be completed.

In Table I are listed some of the values of k obtained from Eq. (29). The beat[10] reported is the one that contains the total transfer instant, while the cycle number refers to the number of primary oscillation cycles (primary current) after which the transfer is complete. Note how this number of cycles is simply $c/2$, while the beat number is given by b [Eq. (30)]. The time instant when all the initial charge has been transferred from the primary to the secondary is (see Fig. 4)

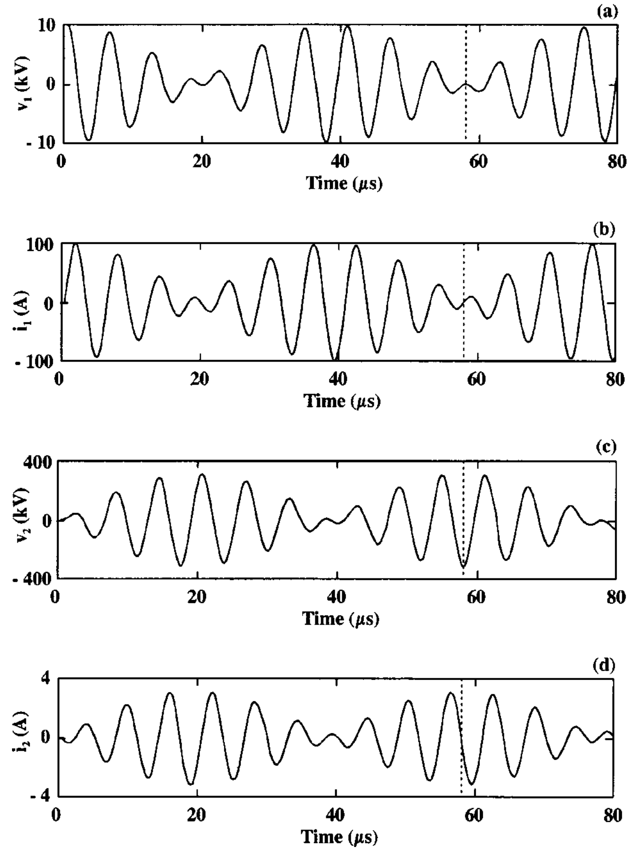

FIG. 4. Voltage and current in a Tesla transformer simulated with $C_1=10$ nF, $L_1=100\ \mu$H, $R_1=0$, $C_2=10$ pF, $L_2=100$ mH, $R_2=0$, and $k=0.161$, and an initial voltage of 10 kV on C_1. All the energy initially on C_1 is transferred to C_2 after 10 primary cycles, at 58 μs; (a) primary voltage, (b) primary current, (c) secondary voltage, and (d) secondary current.

$$v_1=0, \quad v_2=V_2, \quad i_1=0, \quad i_2=0. \tag{31}$$

Under the condition $T=1$, Eq. (3) gives

$$v_2(t)|_{T=1}=V_1\sqrt{\frac{L_2}{L_1}}\sin\left(\frac{w_2+w_1}{2}t\right)\sin\left(\frac{w_2-w_1}{2}t\right). \tag{32}$$

Supposing lossless circuits and $T=1$, varying k does not influence the maximum value of v_2 achievable, as long as Eq. (29) is satisfied.

ACKNOWLEDGMENT

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